The 5th Math Fest. Khon-Kaen, Dec 23, 2023

From Tasks to Problems

Painting Fences and Fermat's Little Theorem

Mathematics should always be well colored!

Little Tasks

How many 3-digit numbers are there in total? **900**

How many 3-digit numbers are there
if all the digits are odd? (e.g. 379 or 911 but not 247) **125**

How many 3-digit numbers are there
if all the digits are even? (e.g. 244 or 806 but not 241) **100**

How many 3-digit numbers are there if all the
digits are different? (e.g. 564 or 805 but not 447) **648**

Combinatorial multiplication rule

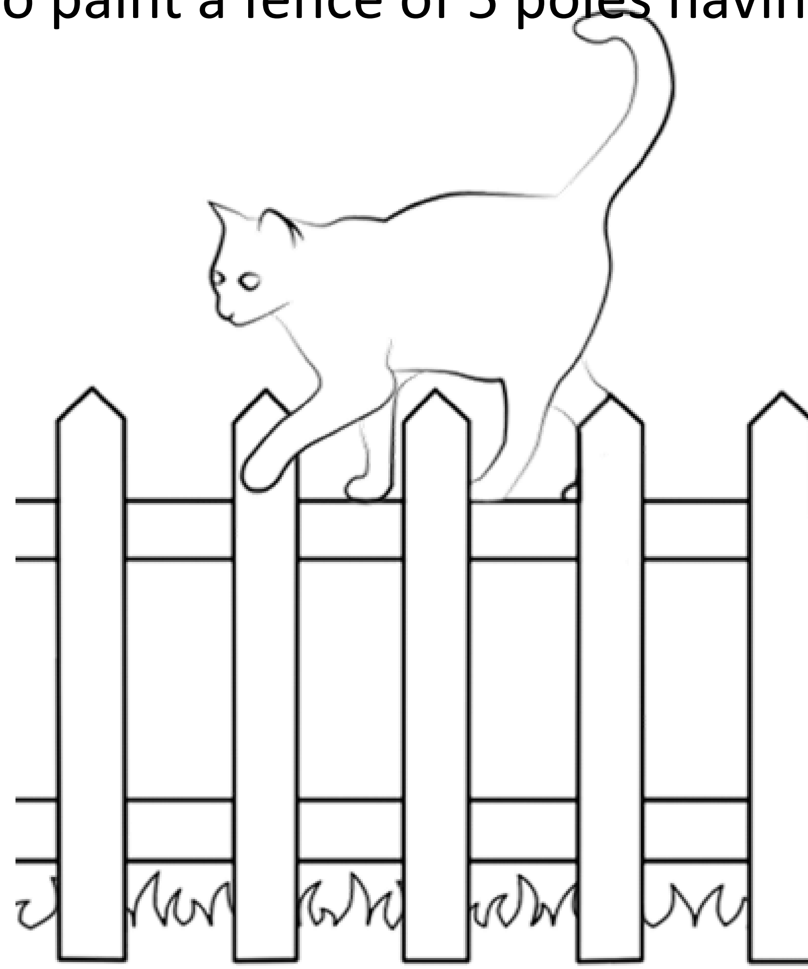
How many ways are there to make a boy-girl pair if there are 5 girls and 5 boys? **25**

How many pairs could be made of those 10 in total? **45**

How many diagonals does a 7-gon have? **14**

Let's paint a fence using the multiplication rule

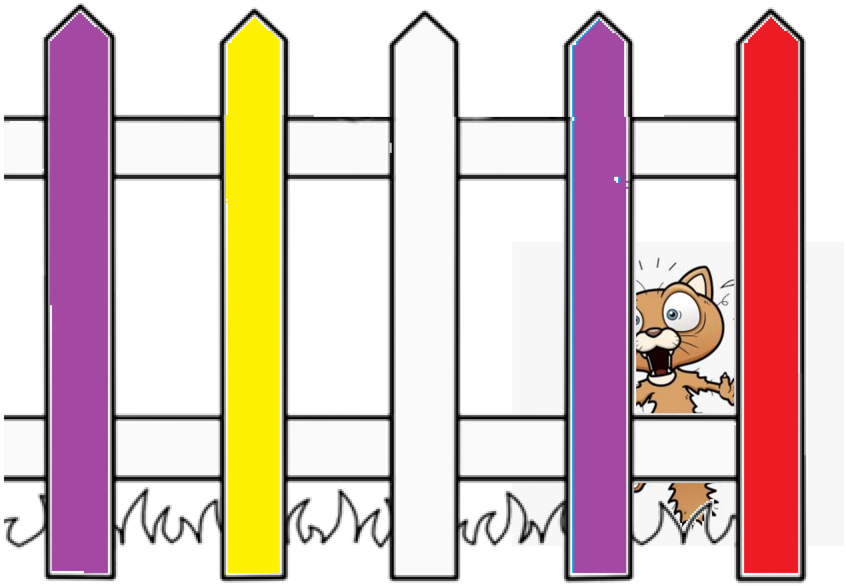
How many ways are there to paint a fence of 5 poles having 6 colors?



Let’s paint a fence using the multiplication rule

How many ways are there to paint a fence of 5 poles having 6 colors?

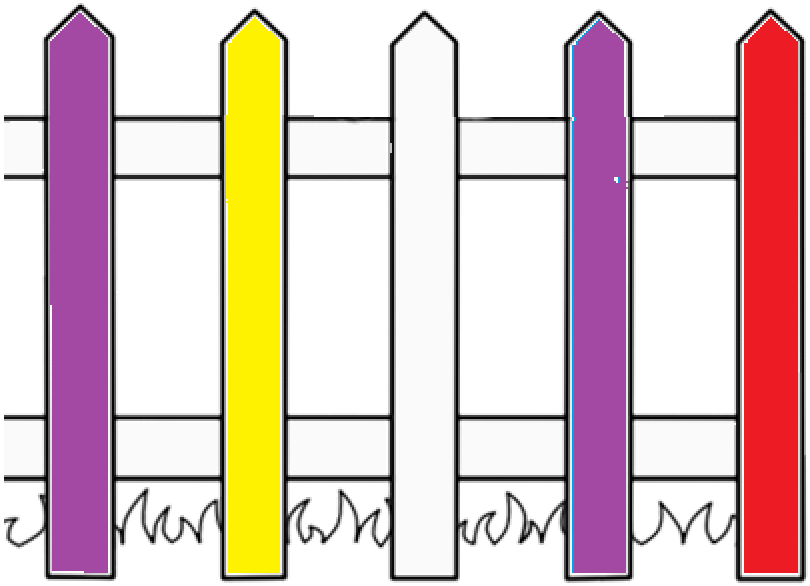
$$6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5 = 7776$$

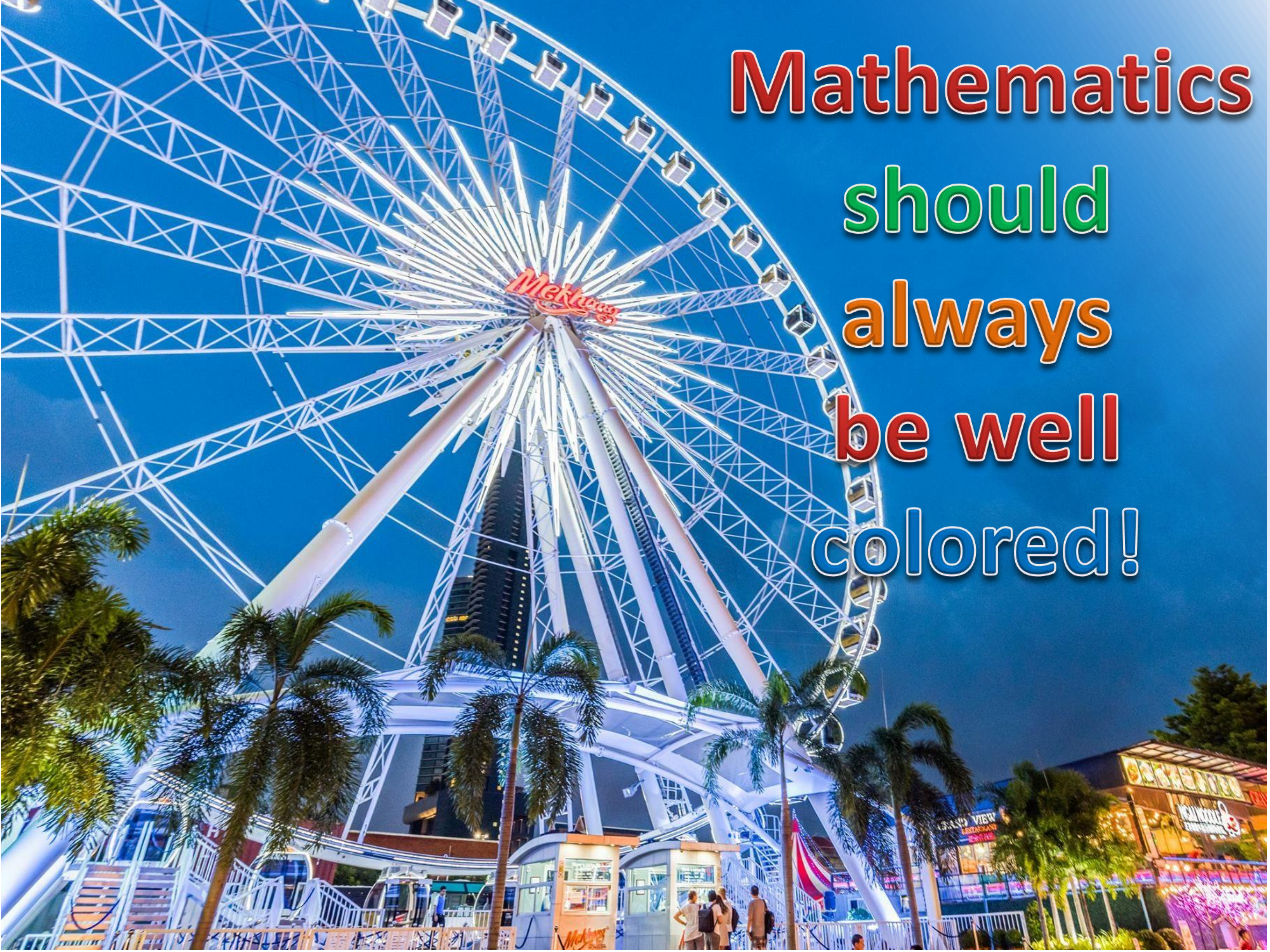


Painting the Fence

How may ways are there to do the same but using at least 2 colors?

$$6^5 - 6 = 7770$$

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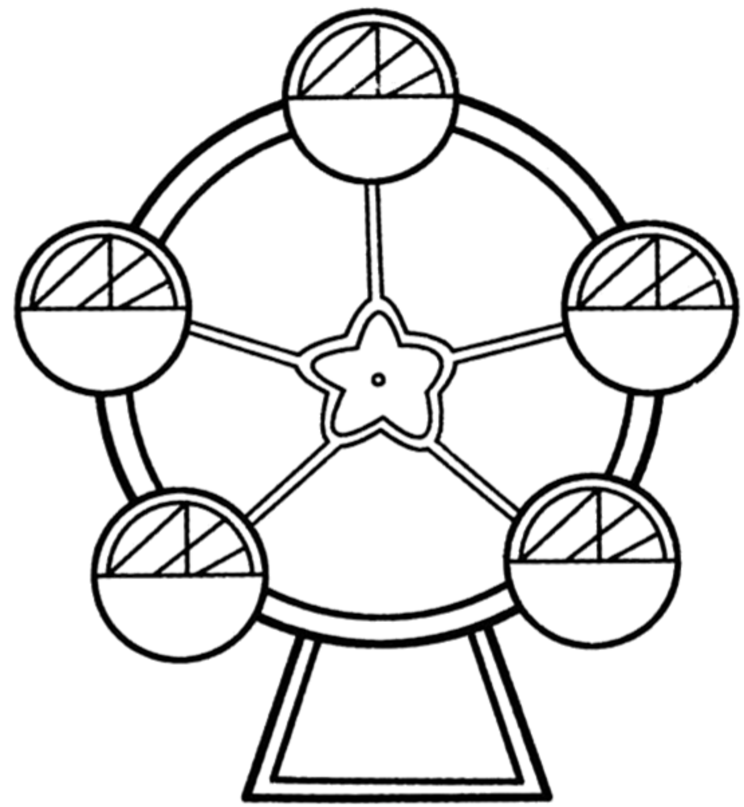
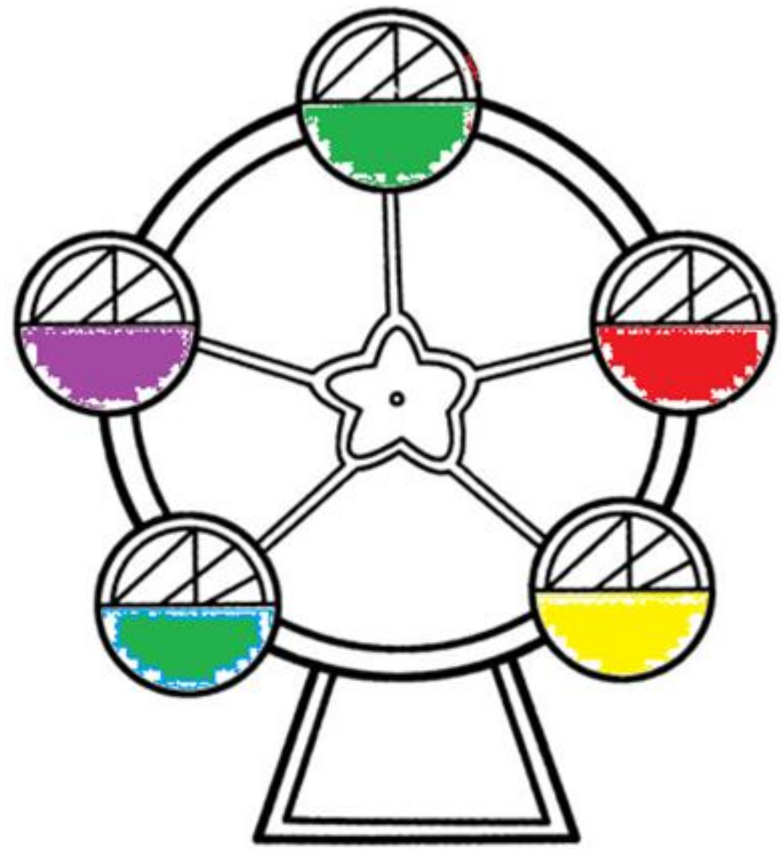
Painting a Ferris Wheel

How many ways are there to paint a Ferris wheel of 5 cabs using at least 2 colors out of 6?



Painting a Ferris Wheel

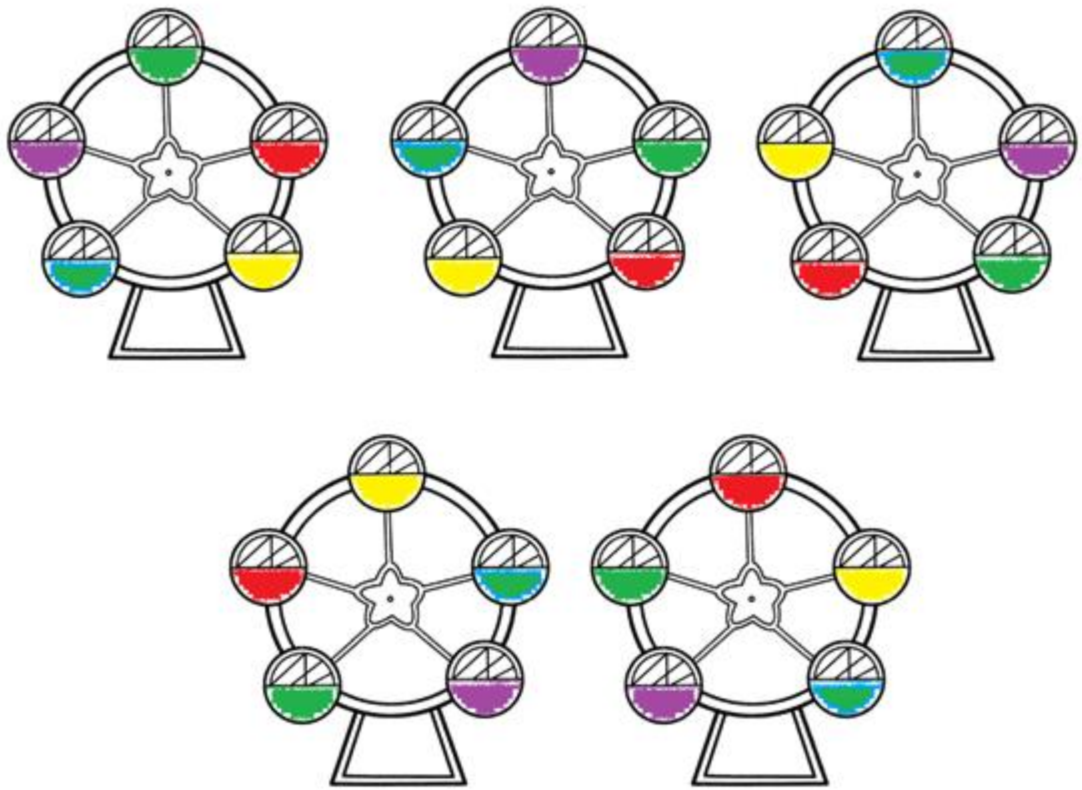


First idea says it should be the same as with the fence:

$$6^5 - 6 = 7770$$

But…

It can rotate!



These paintings are not different. They are the same.
So 5 matching paintings really give only 1 pattern.

It can rotate

So, the correct answer is not $6^5 - 6$ but

$$\frac{6^5 - 6}{5} = \frac{7770}{5} = 1554$$

Pierre de Fermat and his Little Theorem

$$\frac{n^p - n}{p} - \text{whole number ???}$$

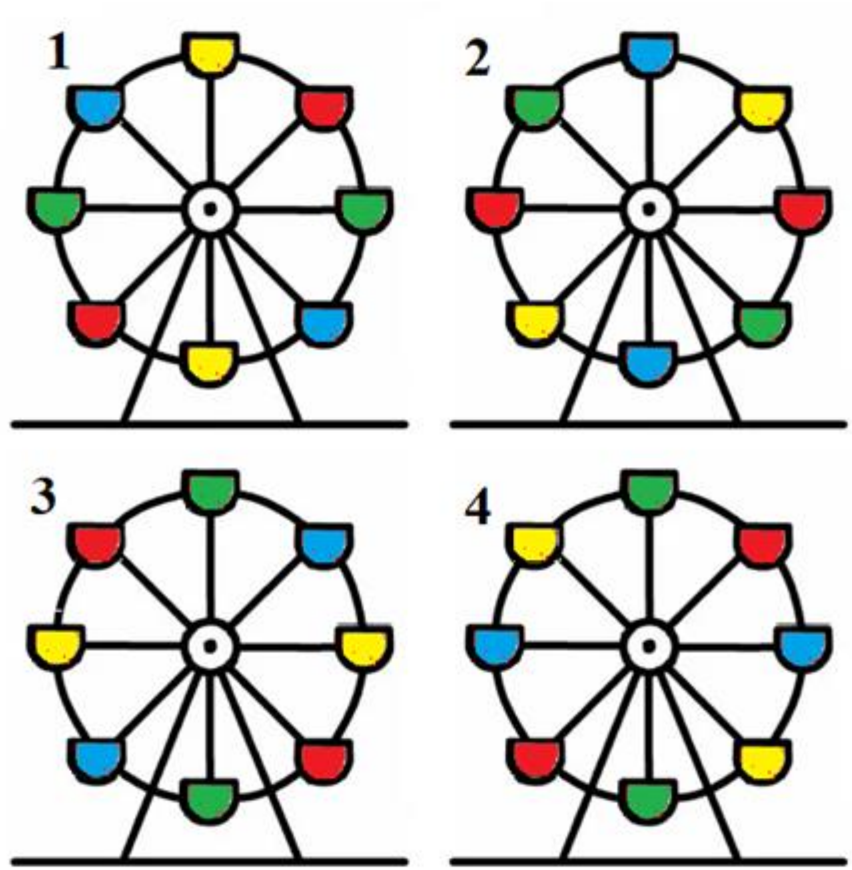
$$\frac{4^3 - 4}{3} = \frac{60}{3} = 20$$

$$\frac{3^4 - 3}{4} = \frac{78}{4} = 19,5$$

p\n	2	3	4	5	6	7	8	9
2	1	3	6	10	15	21	28	36
3	2	8	20	40	70	112	168	240
4	No	No	63	155	No	No	1022	1638
5	6	48	204	624	1554	3360	6552	11808
6	No	121	682	No	7775	19607	No	88572
7	18	312	2340	11160	39990	117648	299592	683280
8	No	No	No	No	No	No	2097151	5380839
9	No	No	No	No	No	No	14913080	43046720
10	No	No	No	976562	6046617	No	No	No
11	186	16104	381300	4438920	32981550	179756976	780903144	2852823600
12	No	No	1398101	No	No	No	No	23535794706
13	630	122640	5162220	93900240	1004668770	7453000800	42288908760	1,95528E+11
14	No	No	No	No	No	48444505203	3,14146E+11	No
15	No	No	71582788	2034505208	31345665638	No	No	1,37261E+13
16	No	No	No	No	No	No	No	1,15814E+14
17	7710	7596480	1010580540	44878791360	9,95686E+11	1,36841E+13	1,32459E+14	9,81011E+14
18	No	No	No	No	No	No	1,0008E+15	8,33859E+15
19	27594	61171656	14467258260	1,00387E+12	3,20716E+13	5,99942E+14	7,58501E+15	7,10975E+16
20	No	No	No	4,76837E+12	1,82808E+14	3,98961E+15	5,76461E+16	6,07883E+17
21	No	No	No	No	1,04462E+15	2,65974E+16	4,39208E+17	5,21043E+18

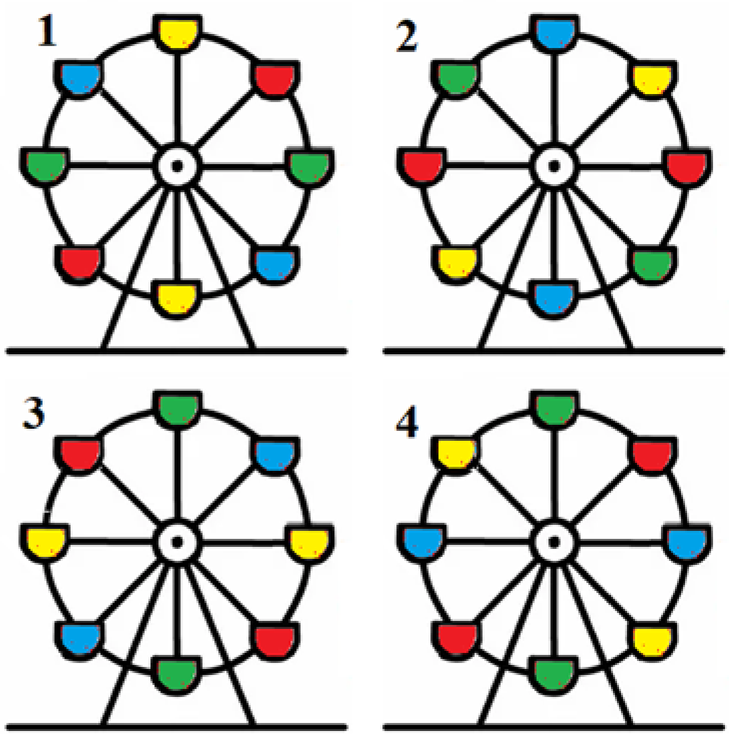
Why prime numbers only?

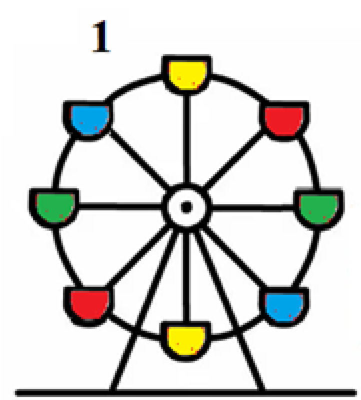
What if p is not prime? Let’s check for p = 8. The whole point is the rotation. Let’s paint the wheel as shown below and begin spinning it. Firstly all goes alright…



Why prime numbers only?

but as we moved the wheel 5th time we see that it coincided with the first painting. So in this case not 8 but only 4 paintings should be put together into one pattern. That’s the problem with non-prime numbers: they have other divisors apart from 1 and itself.





We proved the theorem!

The number of all paintings is exactly

$$\frac{n^p - n}{p}$$

where *n* is a number of available colors
and *p* is a **prime number** of the cabins.

But the number of paintings can't be fractional. It must be whole. The theorem is proved.

Fermat's Little Theorem

If p is a prime number and n – any natural, then

$$n^p - n$$

is divided by p.

If n is not a multiple of p, then

$$n^{p-1} - 1$$

is divided by p.

Pierre de Fermat. 1640.

His solution was long and complicated. We found how to prove it just painting fences and ferris wheels.

Mathematics should always be well colored!

Thank you