MATHEMATICAL THINKING IN JAPANESE CLASSROOMS

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Introduction: personal background and intention

As a young student, I have experienced mathematics classrooms at the elementary and secondary school level and at the university level in Argentina. After that, I have experienced mathematics classrooms as a graduate student in Israel. In both countries, I taught mathematics at junior high school and high schools. In Israel and the USA, I was engaged in curriculum development, teacher education and research on teaching and learning. It is against that variegated, yet all-Western, background that I was exposed to the fascinating microcosms of Japanese mathematics classrooms at the elementary school level. Before I went to CRICED, at Tsukuba University to work with Professor Masami Isoda for four months, I had read research describing lesson study in Japan (e.g. Fernandez and Yoshida, 2004, and other sources) which focused on comparing and contrasting Japanese (and other Asian classrooms) with American classrooms (e.g. Stevenson and Stigler, 1992, Stigler and Hiebert, 1999). I had also watched and analyzed an algebra and a geometry lesson (published by TIMSS in 1999), and on their basis, an in-service workshop for teachers and teachers of teachers was designed and implemented (Arcavi & Schoenfeld, 2006) drawing on ideas from the Teacher Model developed by Schoenfeld (1998). However, I think I was able to fully appreciate the teaching in mathematics classrooms in Japan only through the non-mediated experience (except for the simultaneous translation) of "being there", watching how lessons evolve, following children's work and discussions, talking to teachers and researchers and sensing the common pedagogical and mathematical characteristics of all the lessons I saw.

If we take the statement that "there is nothing more practical than a good theory" (Lewin, 1952, p. 169) and attempt to formulate its symmetrical version, we may propose that "there is nothing more theoretical than a good practice". This may make little sense as stated, however, it may suggest that a exemplary practice can be a powerful source for theorizing, which in turn may help understand the practice, especially a teaching practice. Our field has many learning theories, however there are not so many instructional theories. It is with the intention of contributing to instructional theories, that I would like to briefly share what I have learned from mathematics classrooms in Japan.

Mathematical Thinking – A Japanese view

"Mathematical thinking is the "scholastic ability" we must work hardest to cultivate in arithmetic and mathematics courses… [it] is even more important than knowledge and skill, because it enables to drive the necessary knowledge and skill" (Katagiri, 2006, p.5). Moreover, "mathematical thinking allows for (1) an understanding of the necessity of

using knowledge and skills (2) learning how to learn by oneself, and the attainment of the abilities required for independent learning" (Katagiri, 2006, p.6)
According to this philosophy, advancing mathematical thinking includes the development of:

- '**attitudes**'– intellectual predispositions towards doing mathematics and solving problems, including perspectives on what mathematics and mathematical activity are,
- '**contents**'– concepts, properties, interrelationships, and
- '**methods**'– inductive and deductive reasoning, analogical thinking, generalization, specialization, symbolization.

This rich view of what constitutes mathematical thinking drives teaching and was clearly reflected in all the lessons I saw and analyzed.

In the following, I concentrate on a particular aspect of the teaching practice: "teacher" actions, decision making, lesson crafting and the classroom setting which are aimed at the development, support and encouragement of sound and independent mathematics thinking. In other words, I describe what Japanese teachers actually do, how they do it, in order to engage students in thinking and learning while they are 'doing' mathematics, and how do teachers connect to students "en masse" in a fruitful and unconstrained way? (Lampert, 2001, p. 424).

Characteristics of the lessons

The following are some of the common characteristics of mathematical lessons in elementary school, which I found to be at the core of supporting mathematical thinking, and which are the result of purposeful planning and crafting by the teachers.

Coherence

All the lessons have a "story" – "A good story is highly organized; it has a beginning, a middle, and an end, and it follows a protagonist who meets challenges and resolves problems that arise along the way. Above all, a good story engages the reader's interest in a series of interconnected events, each of which is best understood in the context of the events that precede and follow it" (Stevenson & Stigler, 1992, p. 177). In other words, each lesson evolves and revolves around a central mathematical problem on which students work bringing in their common sense, their knowledge from outside mathematics, their findings from investigating the problem and their ongoing building on the ideas they produce in situ. The teacher leads students to apply knowledge and ideas that emerge during the lesson. The work has a unifying mathematical thread and sometimes the teacher creates an atmosphere of "suspense" around features of the problem, which fuels interest and maintains students engaged and active.
Coherent lessons that pursue central ideas around a meaningful and interesting problem are related to all the three components of mathematical thinking described above: attitudes, contents and methods. Whole lessons which pursue a central problem have the potential to nurture a view of mathematics as a discipline that tackles complex and relevant problems, which take some time to solve and which include attempts that fail and attempts that succeed, alternative approaches, discussion and exchange of ideas. The content involved

in solving a central problem goes beyond the presentation of a skill or a concept, the teacher involves the students in both conceptual understanding and in procedural activities, which are interwoven and are at the service of each other. In the process students propose methods of work and apply different ways of reasoning.

The following is an example of a lesson centered around one problem "the unfolding of the cylinder" (Arcavi, 2001). This is the second of three lessons (allotted by the curriculum) on the unfolding of solids. During this class, the problem is to design models of unfolded cylinders and then to assemble them in order to check that they indeed yield a cylinder. The goals are that students learn interactively (with concrete materials and with other students) about the structural components of the intervening two-dimensional figures, their relative positions, and, in certain cases, the importance of careful planning and measurements. In the process, students exercise their imagination, spatial visualization abilities, and creativity. The lesson opens with the teacher reminding the class of a previous lesson on the unfolding of a tetrahedron, and asks to think about the shape of an unfolded cylinder. After the class worked on the problem for a while, the teacher invites students to share their drawings on the blackboard. The first proposal is the classical: a rectangle and two tangent circles attached to its largest sides (prototypically, the largest sides are the horizontal). The teacher takes the opportunity to analyze the figure with the class, and to make sure students understand and agree on all the details. Thus, by asking several questions, simple but very important issues are raised and discussed, like:

- the two circles (the bases of the cylinder) should be of the same size,
- the two circles should be tangent to the a corresponding pair of parallel sides of the rectangle (and not secant to them),
- the length of the tangent sides should be equal to the circumference of the circles (students recall the number π and the formula for calculating the circumference),
- the length of the other two sides of the rectangle (to which the circles are not attached) are unconstrained (short sides and long sides will yield short or slender cylinders respectively),
- the points of tangency could be anywhere on each of the opposite sides of the rectangle.

Once these issues were discussed, the teacher encourages the class to produce alternative plane models for an unfolded cylinder. The class begins to propose other models including slicing and re-attaching parts of the bases, and many other creative designs, many of which will not fold into a cylinder. At a certain point, the teacher encourages the students to actually cut their designs, attempt to fold them into a cylinder and see if they succeed. In case of failure, students are encouraged to analyse the sources of their erroneous designs. By the end of the class, all the models are displayed.

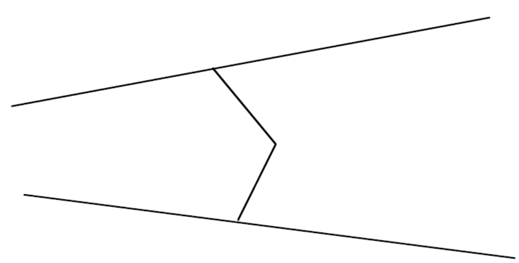
The "story" of this lesson has distinct parts: the discussion of the prototypical unfolding, the planning and design of alternatives, the practical work of assembling of the cylinders out of the models proposed, and the noticing and discussions of the failures. Coherence is not only a characteristic kept within the lesson (including the integration of visual reasoning with calculations, posing conjectures and checking them, analysing failed attempts and discussing each other's solutions), but it is also related to the two other lessons this class has on the unfolding of solids.

Challenging problems

In most of the lessons I saw, there were instances in which I found myself solving mathematical aspects of the problem, as if I were a very engaged student participating in the class. I took this as an indication that the problems and discussions in these lessons are indeed mathematically interesting, challenging, and deep.

Consider, for example, a third grade lesson entitled "New ways of calculation" (for a detailed description see Arcavi 2007), in which the students are asked to calculate a series of multiplications of two numbers between 20 and 30 in which their unit digits add up to 10 (e.g. 23x27, 24x26, 25x25). As the lesson slowly unfolds, the teacher asked students to notice, record and communicate patterns (the way the exercises are handed in does not make the task of finding patterns a straightforward one), to propose an easy algorithm to perform these calculations, and to attempt to explain when and why it works. The new rule students discover and propose is that the result is 600 plus the result of the multiplication of the two digits, e.g. 23x27=600+3x7=621. Obviously, third grade students lack the algebraic tools to generalize and explain why the rule works, thus students work at the edge of their knowledge (or perhaps a little beyond that).

The following example is the central problem of the geometry class of the TIMSS video. "Replace the non-straight boundary dividing two pieces of land in the figure below



by a straight boundary line, while preserving the areas of the original pieces."

I have shared this problem with many knowledgeable mathematics teachers and they worked for a while before finding a solution, agreeing that this is a very difficult problem to be given to 8th graders. Certainly, this challenging problem was given within a coherent sequence of lessons and was supposed to be a non-trivial application of a property studied in a previous lesson: that all the triangles with the same base, and whose third vertex lies on a line parallel to that base, have the same area (the "constant area property").

The above problems are very different in nature, but they share some characteristics: they are not straightforward exercises, they require students to work at the edge of their knowledge, to explore, to discuss different approaches and to slowly device ways to make progress on the basis of mathematical content and different ways of reasoning.

The choice of problems like these implies that Japanese teachers feel very comfortable with their mathematical knowledge. But most importantly, by using these problems for an entire class, teachers enact their confidence that their students can and should engage with mathematical challenges and that they will be able to make progress.

Posing questions

The questions teachers pose to their students during the lessons and their questioning techniques are in consonance with the type of problems that teachers choose to be at the core of each lesson and at the service of solving them and learning from them. Many times the questions request explanations, arguments or counter-arguments. A salient characteristic of the lesson is to make sure that these explanations are fully understood by everybody, and one would expect the following line of questioning to attain this goal: requesting the student to repeat her explanation for the rest of the class, asking the class whether they understood it, asking who agrees or disagrees, and requesting other explanations. These are indeed part of the teacher's repertoire of questions. However, a technique which I found of great interest and importance is none of the above: after a student produces an explanation, an argument or proposes a conjecture the teacher asks the whole class who can explain such explanation or who can tell 'what is the thinking behind such explanation or proposal'. Such request nudges students to carefully learn to listen to each other, and before they can agree or disagree, to take the other's ideas and be able to replay and enact them as if they were theirs. Learning to listen to each other can be highly beneficial in developing the kinds of mathematical thinking the Japanese teachers are after in at least two different directions. Firstly, it may support the nurturing of empathic and caring relationships by conveying the message 'I take your idea and delve deep into it and its merits and sources'. Such a message is the precondition for the development of "academic civility" (Lampert, 2001, p. 431) in a classroom, within which all ideas are respected and valued and inspected mathematically. Secondly, by fully taking the other's perspective, one may be exposed to new ideas and forced to analyze them from within – on the one hand, this helps towards 'decentering' oneself and on the other hand, this may lead to re-inspect one's own knowledge, against the background of what was heard from others (Arcavi & Isoda, 2007). Thus, such a simple request from the teacher may be instrumental in supporting mathematical thinking.

There is another aspect to the questions teachers ask: the redefinition of the role of authority. In my observations, the teachers' authority is reflected in the decision about which task to focus on, which questions to pursue and when, how to distribute the right to speak and how to sequence the activities. The authority is not exerted in determining what is mathematically right or wrong, in this case the teacher deflects, as far as possible, such authority to the mathematics itself (Arcavi et al., 1998), placing a central role on the production of explanations and arguments to settle opposing results. This implies that erroneous answers are not immediately judged as incorrect, they have legitimate status until they are discussed against others. Building on students capacities to evaluate mathematical arguments and ideas places on them responsibility on their own learning and indeed supports the development of mathematical thinking.

Anticipation

Asking good open questions (such as requesting explanations, conjectures and proposals for strategies and ideas) constitutes, in more than one sense, a challenge for the teachers, because when students respond bringing in their proposals and ideas, teachers must do several things at once. Firstly, they have to perform an on site and very quick evaluation of the mathematical merit of the students' proposals, and this implies a solid mathematical background on the part of the teacher and a confidence to put it to use 'online'. Secondly, there must be an evaluation of the pedagogical possibilities that a student proposal affords and a decision regarding how to take advantage of it - this may imply changing the direction of the planned lesson and sometimes even some relinquishing of the control on the new directions the lesson may take. In my view, this is one the most difficult predicaments of the teaching profession. How do Japanese teachers cope with such situations? As far as I understand it, this issue is at the core lesson study: to study a lesson in depth and to implement it several times such that most students' reactions and proposals can be anticipated and only very few are new and surprising. Anticipated student reactions unload from the teacher the burden of on-site decision making. Furthermore, very fruitful student reactions which can contribute to the course of the lesson and which the teacher knows the students can produce them, can be stimulated and looked for at appropriate moments of the lesson.

Diversity

Much has been said about the ethnic (maybe also socio-economical) and cultural homogeneity of the Japanese society. Thus classroom realities are very different from those of the countries I know. Multiculturalism, multilingualism and social deprivation, which are pervasively reflected in classrooms in many Western countries are almost not known in Japan. However, there is another kind of diversity which is as present in Japanese classrooms as in their Western counterparts and which is no less of a challenge to teaching: children differ in their academic achievements and in their mastery of the subject. Japanese teachers cope with such diversity using different pedagogical approaches to a same topic, they have a proper pace and they harness all students' responses from the less to the more sophisticated (in that order) in the development of a lesson. However, no matter how competitive this society may be regarded by many, elementary schools do not track students and the teachers attend to all children in an impressively inclusive way. It is true that the respectful way in which Japanese people treat and address each other in ingrained in the culture and is the common and natural behavior: in contrast to what I have seen in many classrooms in other countries, I have not seen any Japanese teacher raising their voices or reprimanding students. Even when initial commotion was caused by the presence of visitors or by the mere playfulness of the children, the unrest quiets down by itself, mostly without the need of any intervention. Thus, the atmosphere teachers manage to induce in their classrooms is very propitious for thinking, working, asking any kinds of questions and freely expressing thoughts and ideas by everybody.

Pace

Some of my colleagues with whom I shared videos of Japanese lessons pointed to me that in many instances during the lesson they felt that the pace was slow. Interestingly, this is also the impression of many others (Stevenson & Stigler, p. 194). The slow pace of the lesson is related to its coherence (slowly building the "story" as described above) and to the teacher intention to be as inclusive as possible and to leave nobody behind. Moreover, this pace is a reflection of another deep belief: thinking takes time, ideas need to be mulled over, applied, discussed and approached from different directions, and if this is to be taken seriously there is no room for rushing.

Setting and devices

The 'architectural' setting of the classroom is traditional: lines of benches in rectangular rooms with a board at the front, simple teaching aids (such as magnetic manipulatives, paper cutting and the like). This setting does not prevent students from working with peers, come in groups to the board, and even moving around when the teacher thinks it is appropriate.

The blackboard plays a very central role in all classrooms, is not only a working space, but an organizational device, a thinking tool and a medium to record the flow of the lesson and its main ideas. Many times at the end of the class, if one looks at the board, it can tell the whole "story" of the lesson, especially displaying students' work and their differences of approaches.

Empathy

It is my impression that all students are affectively "contained" within a solid support net: teachers and parents work hard to closely follow up each of the students and attend to their needs as they arise. Teachers treat all their students with respect, they allow them to be boisterous and at times they even promote that. Some teachers also display their sense of humor, making the whole atmosphere of the class agreeable and supportive. Empathy seems to be yet another teaching strategy, which does not come at the expense of being intellectually demanding. Empathy seems to be characteristic of the way teachers address each other when discussing lesson plans and criticizing lessons. Teachers are used to expose their teaching to colleagues knowing that the analysis of their moves will be deep and thorough but very respectful and aimed at learning from each other.

The above characteristics are very different from each other. Some refer to a very deep pedagogical idea, some merely describe the physical setting, some present mathematical features, others refer to inter-human relationships. I would like to claim that maybe the uniqueness of the Japanese classrooms is due to the synergy of all these characteristics and to the professionalism with which of them each of them is treated.

Open questions

A first surprise regarding mathematics classrooms refers to something I did not see in them: computerized technologies. In spite of the many innovative proposals (and studies of their feasibility) about ways to introduce computerized and communication technologies both in the Japanese academia as well as in the academia of Western countries, I have not seen the use of computers in Japanese classrooms. In my view, the work with computerized technologies could fit in with the Japanese characterization of what mathematical thinking should be supported, and its availability in Japan is not an issue – thus, I wonder why it has not entered the classroom.

A second surprise refers to the shift in pedagogical practices that occur in secondary school, where most of the lessons consist of teacher lectures.

And finally, I wondered a lot about the existence, proliferation and success of "juku" schools (and possibly other out of school activities in mathematics), which are attended by a large number of students and are taken so much for granted by Japanese society at large, and which mostly emphasize drill and practice. This phenomenon can have several underlying reasons. For example, is it assumed and agreed that the time devoted by schools is insufficient, and drill should be learned elsewhere? Or, is it assumed that students need "extra practice"? Or, students should not have that much free time after school and their learning must be extended beyond the formal schooling? Or, students' full potential cannot be completely developed by the school only? Or, should students meet other teaching styles? Or….

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